Algebraic Torus

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$\mathrm{GL}_1/\mathbb{C}$ is a split torus. Consider the field extension $\mathbb{C}/\mathbb{R}$. Then $\mathbb{C}$ has the inner product given by

$$\langle z, z' \rangle \quad := \quad \bar{z}z'.$$

We can look at the elements of $\mathbb{C}$ that preserve this inner product,

$$\begin{aligned} U(1) \quad &:= \quad \{c \in \mathrm{GL}_1(\mathbb{C}) : \forall z, z' \in \mathbb{C}, \quad \langle cz, cz' \rangle = \overline{cz}cz' = \bar{z}z'\} \\ &= \{c \in \mathrm{GL}_1(\mathbb{C}) : |c| = 1\}. \end{aligned}$$

Note that this is a (real) line topologically so we do not expect it to be a complex variety. Indeed this defines a **real** algebraic group given by the zero locus in $\mathbb{R}^2$ of the two variable polynomial $x^2 + y^2 - 1$. In other words,

$$U(1) \cong \mathrm{MaxSpec}\big(\mathbb{R}[x, y]/(x^2 + y^2 - 1)\big).$$

Now if we base change to $\mathbb{C}$ we have

$$\begin{aligned} \mathbb{R}[x, y]/(x^2 + y^2 - 1) \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C}[x, y]/\big((x + iy)(x - iy) - 1\big) \\ &\cong \mathbb{C}[s, t]/(st - 1) \\ &\cong \mathbb{C}^*. \end{aligned}$$

Thus $\mathrm{GL}_1/\mathbb{C}$ is the complexification of the torus $U(1)$.